Notes on the Neumann Boundary Condition and Periodic Boundary Condition when using the FD method

February 19, 2019

1 Neumann Boundary Condition

After talking with Yuan Hu, I realized the method dealing with Neumann boundary condition.

1.1 Problem

In order to deal with the neumann boundary condition using the finite difference scheme, we need some tricks.

The easiest way to do this is to use the first order approximation, but this is not good enough.

One example is:

We discrete the domain into grids, and the domain we want to discrete is one dimensional. Basically we want to set the left hand side as the neumann boundary condition and that is we have:

$$\left.\frac{\partial \phi}{\partial x}\right|_{x=0} = \alpha \tag{1}$$

$$\left.\frac{\partial \phi}{\partial x}\right|_{x=L} = \beta \tag{2}$$

In order to approximate this, we have:

$$\frac{\phi_1 - \phi_0}{x_1 - x_0} = \frac{\phi_1 - \phi_0}{h} = \alpha \tag{3}$$

$$\frac{\phi_i - \phi_{i-1}}{x_i - x_{i-1}} = \frac{\phi_i - \phi_{i-1}}{h} = \beta \tag{4}$$

However, this might not be a good approximation. Since we may need a lot of points to make the solution converge to the exact one.

1.2 Solution 1

Consider there is a ghost point, ϕ_{-1}. In this way we have that:

$$-\frac{\phi_{-1} - 2\phi_0 + \phi_1}{h^2} = \rho \tag{5}$$

$$\frac{\phi_1 - \phi_{-1}}{2h} = \alpha \tag{6}$$

We can immediately get:

$$-\frac{\phi_1 - \phi_0}{h^2} = -\frac{\alpha}{h} + 0.5 * \rho_0 \tag{7}$$

The only difference is that we can consider the effects of the boundary's data inside the equation.

Consider there is a ghost point, ϕ_{i+1}. In this way we have that:

$$-\frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{h^2} = \rho \tag{8}$$

$$\frac{\phi_{i+1} - \phi_{i-1}}{2h} = \beta \tag{9}$$

We can immediately get:

$$\frac{\phi_i - \phi_{i-1}}{h^2} = \frac{\beta}{h} + 0.5 * \rho_i \tag{10}$$

The only difference is that we can consider the effects of the boundary's data inside the equation.

1.3 Solution 2

Only use the inside points to found the coefficients matrix or use the one side 2nd order approximation for the neumann boundary condition.

2 Periodic Boundary Condition

For the Periodic boundary condition problem, we should use the matrix form to solve it. However, if we write the matrix directly, we may face one problem: The Matrix can not be inverted. To deal with this, we can substitute the last line of the matrix into:

$$\phi_0 + \phi_1 + \phi_2 ... + \phi_{N-1} = 0 \tag{11}$$

In this way, the matrix can be solved using the LU decomposition. However, if we use this matrix, the LU process will be very slow. So we can do this in an alternative way.

We use this equation for the last row of the matrix:

$$\phi_N - \phi_0 = 0 \tag{12}$$

However, if we use this condition, there will be a shift on the amplitude, though the shape of the function will remain same in approximation. To deal with this, we need to post-process the solution we got by doing this:

$$x_i = x_i - \bar{x} \tag{13}$$

for all i.

In this way we can ensure the equation 6.

$$\phi_0 + \phi_1 + \phi_2 ... + \phi_{N-1} = 0 \tag{14}$$