The Velocity Distribution Function.

1. Phase Space.

Consisted of $\vec{x}$. $\vec{v}$ space

The instantaneous state of a particle is described by a point in the phase space.

number density :

$$n(\vec{x}, t) = \lim_{\Delta V \to 0} \frac{\Delta N}{\Delta V}$$

$$N = \int_V n(\vec{x}, t)\, d^3x$$

$$\therefore \quad N = \int_V \int_{-\infty}^{\infty} f(\vec{x}, \vec{v}, t)\, d^3x\, d^3v$$

$$\therefore \quad n(\vec{x}, t) = \int_{-\infty}^{\infty} f(\vec{x}, \vec{v}, t)\, d^3v$$

Moment of the physics quantities.

$$\langle \alpha(\vec{x}, \vec{v}, t) \rangle = \frac{1}{n(\vec{x}, t)} \int_{-\infty}^{\infty} \alpha f(\vec{x}, \vec{v}, t)\, d^3v$$

Average velocity :

$$\vec{U} = \langle \vec{v} \rangle = \frac{1}{n} \int_{-\infty}^{\infty} \vec{v} f(\vec{x}, \vec{v}, t)\, d^3v$$

Kinetic energy density.

$$W = \frac{1}{2} n m \langle v^2 \rangle = \int_{-\infty}^{\infty} \frac{1}{2} m v^2 f(\vec{x}, \vec{v}, t)\, d^3v$$

Pressure tensor

$$\overrightarrow{P} = nm\langle \vec{v}\vec{v}\rangle = \int_{-\infty}^{\infty} m(\vec{v}-\vec{U})(\vec{v}-\vec{U})\, f(\vec{x},\vec{v},t)\, d^3v$$

Random Velocity $\quad \vec{c} = \vec{v} - \vec{U}$.

$$\langle c\rangle = 0$$

The mean kinetic energy.

$$\langle E\rangle = \frac{1}{2} m \langle c^2\rangle$$

Temperature of the particles.

$$KT_j = \frac{1}{2} m \langle c_j^2\rangle$$

$$KT = \frac{1}{2} m\langle c_x^2\rangle = \frac{1}{2} m\langle c_y^2\rangle = \frac{1}{2} m\langle c_z^2\rangle \; ; \quad \langle E\rangle = \frac{3}{2}KT$$

Scalar pressure:

$$p = nm\frac{\langle c^2\rangle}{3} = nKT$$

The Maxwellian Distribution Function:

$$f_M(v) = n\left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}} \exp\left(-\frac{m(\vec{v}-\vec{U})^2}{2kT}\right)$$

When $\vec{U}=0$, the distribution is simply called the Maxwellian distribution. When $\vec{U}\neq 0$, the distribution is called Drifting Maxwellian Distribution.

Local Maxwellian distribution.

$$f_M(\vec{x},\vec{v},t) = n(\vec{x},t)\left(\frac{m}{2\pi KT(\vec{x},t)}\right)^{\frac{3}{2}} \exp\left(-\frac{m(\vec{v}-U(\vec{x},t))^2}{2kT(\vec{x},t)}\right).$$

Normalized Maxwellian Distribution.

$$\hat{f}_M = \frac{f_M}{n} = \frac{1}{(\sqrt{\pi}\, v_{th})^3} \exp\left(-\frac{(\vec{v}-\vec{U})^2}{v_{th}^2}\right), \quad v_{th} = \sqrt{\frac{2kT}{m}}$$

If $U = 0$, then

$$\hat{f}_M = \hat{g}_M(v_x)\, \hat{g}_m(v_y)\, \hat{g}_m(v_z)$$

Gaussian Distribution:

$$\hat{g}_M(v_x) = \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} \exp\left(-\frac{m v_x^2}{2kT}\right) = \frac{1}{\sqrt{\pi}\, v_{th}} \exp\left(-\frac{v_x^2}{v_{th}^2}\right)$$

$$\langle v_x \rangle = 0$$

$$\langle v_x^2 \rangle = \frac{kT_x}{m}$$

$$\langle v_x^2 \rangle^{\frac{1}{2}} = \sqrt{\frac{kT}{m}}$$ ⇒ Root-Mean-Square 1-D Velocity.

$$\langle |v_x| \rangle = \frac{1}{n}\int_{-\infty}^{\infty} |v_x|\, g_m\, dv_x = \sqrt{\frac{2kT}{\pi m}}$$ ⇒ 1-D speed.

3. One-sided. Average Velocity and Flux.

- Stationary Gas.

$$\langle v_x \rangle^+ = \int_{v_x=0}^{\infty} v_x\, \hat{g}_M(v_x)\, dv_x \int_{v_y=-\infty}^{\infty} \hat{g}_M(v_y)\, dv_y \int_{v_z=-\infty}^{\infty} \hat{g}_M(v_z)\, dv_z$$

$$= \frac{1}{2}\langle |v_x| \rangle = \frac{v_{th}}{2\sqrt{\pi}} = \langle v_x \rangle^-$$

$$\Gamma_x = \frac{1}{2} n \langle |v_x| \rangle = \frac{1}{2} n \cdot \sqrt{\frac{2kT}{\pi m}}$$

- Drifting Gas:

$$\langle v_x \rangle_+ = \frac{1}{2}\left[\frac{v_{th}}{\sqrt{\pi}} \exp\left(-\frac{v_d^2}{v_{th}^2}\right) + v_d\left(1 + \mathrm{erf}\left(\frac{v_d}{v_{th}}\right)\right)\right]$$

$$\langle v_x \rangle_- = \frac{1}{2}\left[-\frac{v_{th}}{\sqrt{\pi}} \exp\left(-\frac{v_d^2}{v_{th}^2}\right) + v_d\, \mathrm{erfc}\left(\frac{v_d}{v_{th}}\right)\right]$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$$

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-x^2} dx$$

$\mathrm{erf}(0) = 0.$ $\mathrm{erfc}(0) = 1$ $\mathrm{erf}(\infty) = 1,$ $\mathrm{erfc}(\infty) = 0$

When $\frac{v_d}{v_{th}} = 0$, we go back to the stationary case.

when $\frac{v_d}{v_{th}} = \infty$, $\langle v_x \rangle^+ = v_d.$ $\langle v_x \rangle_- = 0.$

$\boxed{\text{The cold Beam case}}$

÷ Some special case.

- Momentum Flux. ($Q = nm v_x \cdot v_x$)

$$P_x = n \int_0^\infty m v_x \cdot v_x \hat{g}_m(v_x) \, dv_x \int_{-\infty}^{+\infty} \hat{g}_m(v_y) \, dv_y \int_{-\infty}^{+\infty} \hat{g}_m(v_z) \, dv_z$$

$$= \frac{nm}{2} \left[\frac{v_{th}}{\sqrt{\pi}} v_d \exp\left(-\frac{v_d^2}{v_{th}^2}\right) + v_{th}^2 \left[1 + \mathrm{erf}\left(\frac{v_d}{v_{th}}\right) \right] \left[\frac{1}{2} + \left(\frac{v_d}{v_{th}}\right)^2 \right] \right]$$

- Surface stress. ($Q = nm v_x \cdot v_y$).

$$P_y = n \int_0^\infty v_x \cdot \hat{g}_m(v_x) \, dv_x \int_{-\infty}^{+\infty} v_y \hat{g}_m(v_y) \, dv_y \int_{-\infty}^{+\infty} \hat{g}_m(v_z) \, dv_z$$

$$= \frac{nm}{2} v_d \sin\theta \left[\frac{v_{th}}{\sqrt{\pi}} \exp\left(-\frac{v_d^2 \cos\theta^2}{v_{th}^2}\right) + \frac{v_d \cos\theta}{v_{th}} \left(1 + \mathrm{erf}\left(\frac{v_d \cos\theta}{v_{th}}\right)\right) \right]$$

- Heat Flux: $Q = \frac{1}{2} m (v_x^2 + v_y^2 + v_z^2) \cdot n v_x$

$$q_{i,tr} = \frac{nm}{4} v_{th}^2 \left[\frac{1}{\sqrt{\pi}} \left[\frac{v_d^2}{v_{th}} + 2 v_{th} \right] \exp\left(-\frac{v_d^2}{v_{th}^2}\right) + v_d \left(\frac{v_d^2}{v_{th}^2} + \frac{5}{2}\right) \left\{ 1 + \mathrm{erf}\left(\frac{v_d}{v_{th}}\right) \right\} \right]$$

If $v_d = 0$, $\Rightarrow$ $q_{i,tr} = \frac{nm}{2\sqrt{\pi}} v_{th}^3$

- Internal Energy Flux.

$$e_{int} = \frac{\xi}{2} kT \qquad \alpha = n v_x \cdot \frac{\xi}{2} kT$$

$$q_{i,in} = \frac{nm}{8} v_{th}^2 \left(\frac{5-3\gamma}{\gamma-1} \right) \cdot \left[\frac{v_{th}}{\sqrt{\pi}} \exp\left(-\frac{v_d^2}{v_{th}^2}\right) + \frac{v_d}{v_{th}} \left\{ 1 + \mathrm{erf}\left(\frac{v_d}{v_{th}}\right) \right\} \right]$$

Total Energy Flux:

$$q = q_{i,tr} + q_{i,in}.$$

Flat Surface in Collisionless Unmagnetized Plasmas

- ion & electron flux to the surface
- Sheath structure
- Floating potential of surface.

Surface in Stationary, Thermal. Collisionless Plasma

This means no drifting velocity, the ions & electrons move to surface due to the thermal motion.

In this way, at the place far away from the surface, $x = -\infty$, we get that

From distribution functions.

$$\Gamma_e(-\infty) = n_0 \sqrt{\frac{kT_e}{2\pi m_e}}$$

$$\Gamma_i(-\infty) = n_0 \sqrt{\frac{kT_i}{2\pi m_i}}$$

$$\Rightarrow \quad \frac{\Gamma_e}{\Gamma_i} = \sqrt{\frac{T_e}{T_i} \cdot \frac{m_i}{m_e}}$$

See, as T_i and T_e basically in the same order, and $m_i >$ m_e, Γ_e is much larger than Γ_i.

This means the surface will establish a negative potential first. Then a equilibrium will be achieved.

At steady state there should be.

$$\Gamma_e(0) = \Gamma_i(0)$$

- Electron flux to the surface.

From Energy conservation:

$$\frac{1}{2} m_e v_x^2(-\infty) = \frac{1}{2} m_e v_x^2(0) - e\phi_w.$$

Only the electrons with velocity bigger than $\sqrt{\frac{-2e\phi_w}{m_e}}$ can reach the surface.

$$\therefore \quad \Gamma_{e+} = \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{\infty} dv_y \int_{\sqrt{\frac{-2e\phi_w}{m_e}}}^{\infty} v_x f_e \, dv_x$$

Since we assume the initial distribution is a Maxwellian Distribution

$$\therefore \quad \Gamma_{e+}(-\infty) = n_0 \sqrt{\frac{kT_e}{2\pi m_e}} \exp\left(\frac{e\phi_w}{kT_e}\right) = \Gamma_e(0), \quad \phi_w \leq 0$$

If $\phi_w > 0$, then

$$\Gamma_{e+}(-\infty) = n_0 \sqrt{\frac{kT_e}{2\pi m_e}} = \Gamma_e(0), \quad \phi_w > 0$$

The equations are valid under condition that $|\phi_w|$ is not significantly larger than T_e.

- Ion flux to the surface.

Similarly, we can get:

$$\Gamma_{i+}(-\infty) = n_0 \sqrt{\frac{kT_i}{2\pi m_i}} = \Gamma_i(0), \quad \phi_w \leq 0$$

$$\Gamma_{i+}(-\infty) = n_0 \sqrt{\frac{kT_i}{2\pi m_i}} \exp\left(-\frac{e\phi_w}{kT_e}\right) = \Gamma_i(0) \quad \phi_w > 0$$

At steady state we may get.

$$\Gamma_e(0) = \Gamma_i(0).$$

$$n_0 \sqrt{\frac{kT_e}{2\pi m_e}} \exp\left(\frac{e\phi_w}{kT_e}\right) = n_0 \sqrt{\frac{kT_i}{2\pi m_i}}$$

$$\therefore \quad \phi_w = -\frac{kT_e}{e} \ln \sqrt{\frac{m_i}{m_e} \cdot \frac{T_e}{T_i}}$$

If, $T_e \sim T_i$, $m_i/m_e = 1836$, $\phi_w \sim 3.75 \cdot \frac{kT_e}{e}$.

For Surface in Drifting, Thermal, Collisionless plasma, simply use the drifting Maxwellian Distribution

Surface in Cold (cold ion, thermal electron). collisionless Plasma.

if $\left|\frac{e\phi}{kT}\right| \gg 1$, then that component of plasma can be seen as "cold". In this way the process will become a non-linear process. [Easy to explain. if the ϕ is big enough, then it can magnificently influence the components far away.

consider cold ion. thermal electron., assume $T_i \sim 0$

$$\varepsilon_0 \cdot \frac{d^2\phi}{dx^2} = -e\left(n_i(x) - n_e(x)\right), \quad n_e(x) = n_0 \exp\left(\frac{e\phi}{kT_e}\right)$$

For the ions. we must use that

$$n_0 u_0 = n_i(x)\, u_i(x)$$

$$\frac{1}{2} m_i u_0^2 = \frac{1}{2} m_i u_i^2(x) + e\phi_w(x).$$

This will lead to.

$$n_i(x) = \frac{n_0}{\sqrt{1 - \frac{2e\phi}{m_i u_0^2}}}$$

Hence.

$$\varepsilon_0 \cdot \frac{d^2\phi}{dx^2} = e n_0 \left(\exp\left(\frac{e\phi}{kT_e}\right) - \frac{1}{\sqrt{1 - \frac{2e\phi}{m_i u_0^2}}} \right).$$

Normalize it.

$$\hat{\phi} = -\frac{e\phi}{kT_e}, \quad \hat{x} = \frac{x}{\sqrt{\frac{n_0 e^2}{\varepsilon_0 kT_e}}}, \quad M = \frac{u_0}{\sqrt{kT_e / m_i}}$$

Define a ion acoustic velocity.

$$C_s = \sqrt{\frac{kT_e}{m_i}}$$

$$\therefore \quad \hat{\phi}'' = \frac{1}{\sqrt{1 + \frac{2\hat{\phi}}{M^2}}} - \exp(-\hat{\phi})$$

As the Bohm crition said, ~~we must~~ the ions must reach proper velocity when it goes into the sheath.

Then there must be some ambient E field exists. this is called pre-sheath.

~~As a~~ If the ions are stationary at first, then the enter velocity should be. C_s. (ion acoustic velocity).

If the ions has a drifting velocity much larger than the thermal velociny then the enery velocity should be the drifting velocity.

According to the Child-Langmuir's law, the ion current density to the surface is

$$J_i = e n_0 U_0 = \frac{4}{9}\sqrt{\frac{2e}{m_i}} \cdot \frac{\varepsilon_0 |\phi_w|^{\frac{3}{2}}}{d^2}$$ child-Langmuir Law.

"cold" ion has some ~~assumptions~~ applications in the real-world engineering Problems. For a spacecraft in LEO or in Solar Wind. the cold ion, warm electron assumption will be valid.

In a stationary plasma, where is the Boundary?

Define the sheath boundary is the location divides quasi-neutral region from the non-neutral region.

$\therefore$ $n_i(x=0) = n_e(x=0) = n_{edge}$, not necessarily $= n_0$.

$\phi(x=0) = \phi_{edge}$, $\frac{\partial n_i}{\partial x}\Big|_{x=0} = \frac{\partial n_e}{\partial x}\Big|_{x=0}$

Inside the sheath,

$$n = n_0 \exp\left(\frac{e\phi}{kT_e}\right)$$

$$\therefore \frac{\partial}{\partial x} n_i \Big|_{x=0} = \frac{\partial}{\partial x} n_e \Big|_{x=0}$$

$$\Rightarrow -\frac{n_i(x=0)}{2\phi(x=0)} = n_e(x=0)\cdot\frac{e}{kT_e}, \quad n_i = n_e \text{ at } x=0$$

$$\therefore \phi_{edge} \doteqdot -\frac{kT_e}{2e}$$

$$\therefore n_{edge} = n_0 \exp\left(-\frac{1}{2}\right).$$

From $\frac{1}{2} m_i u_i^2 + e\phi = 0$, $\quad u_i(x=0) = \sqrt{\frac{-2e\phi_{edge}}{m_i}}$

$$\therefore u_i(x=0) = u_{edge} = \sqrt{\frac{kT_e}{m_i}}$$

$\therefore$ In a stationary plasma.

$$J_i = e n_0 \sqrt{\frac{kT_e}{m_i}} \exp\left(-\frac{1}{2}\right).$$

at sheath boundary, the ions have already acquired a velocity of $u_{edge} = \sqrt{\frac{kT_e}{m_i}} = C_s$ or $M=1$

Above are the results considering pre-sheath.

Lecture note.

Plasma Dynamics

- Generalized Ohm's law.
- 1-D MHD Channel Flow.

- Derivation of fluid equations.

First, notify the total derivative of f:

$$\frac{Df}{Dt}. \qquad (1)$$

This can be changed into (transformed into):

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \vec{v}\cdot\nabla_x f + \vec{a}\cdot\nabla_v f \qquad (2)$$

Then think about this system, if there is no pertubation, then

$$\frac{Df}{Dt} = 0 \Rightarrow \frac{\partial f}{\partial t} + \vec{v}\cdot\nabla_x f + \vec{a}\cdot\nabla_v f = 0. \quad \text{Vlasov Equation.} \qquad (3)$$

If: there are some pertubations, (collisions), then this should be:

$$\frac{Df}{Dt} = \left(\frac{\delta f}{\delta t}\right)_c \Rightarrow \frac{\partial f}{\partial t} + \vec{v}\cdot\nabla_x f + \vec{a}\cdot\nabla_v f = \left(\frac{\delta f}{\delta t}\right)_c, \quad \text{Boltzmann Equation} \qquad (4)$$

We then go to the derivation of fluid model.

First, integrate it over v ~~(volume)~~ (velocity).

$$\underset{(1)}{\int \frac{\partial f}{\partial t} d\vec{v}} + \underset{(2)}{\int \vec{v}\cdot\nabla_x f d\vec{v}} + \frac{q}{m}\int (\underset{(3)}{\vec{E}} + \underset{(4)}{\vec{v}\times\vec{B}})\cdot\frac{\partial f}{\partial\vec{v}} d\vec{v} = \underset{(5)}{\int \left(\frac{\partial f}{\partial t}\right)_c d\vec{v}}.$$

We can get:

(1): $\int \frac{\partial f}{\partial t} d\vec{v} = \frac{\partial}{\partial t}\int f d\vec{v} = \frac{\partial n}{\partial t}$ (2): $\int \vec{v}\cdot\nabla_x f d\vec{v} = \nabla_x \int \vec{v} f d\vec{v} = \nabla\cdot(n\vec{u})$

(3): $$\int \vec{E}\cdot\frac{\partial f}{\partial\vec{v}} d\vec{v} = \int \nabla_v\cdot(f\vec{E}) d\vec{v} = \int_\infty (f\vec{E})\cdot d\vec{S} \qquad (5)$$

due to $f\to 0$ faster than. $\vec{E}\to 0$ ($\vec{E}\sim\frac{1}{v^2}$). So (3) vanishes

(4). equals to 0 as $f\to 0$ faster than any power of $\vec{v}$.

(5). is definitely equal to 0.

At last we get the 0-th moment of Boltzmann Equation.

In this way we can also get 1st moment of Boltzmann Equation. (momentuum, equation): See F.F. Chen's textbook, Chapter 7.3. (P. 223) for details:

$$mn\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right] = qn(\vec{E} + \vec{u}\times\vec{B}) - \underbrace{\nabla\cdot\vec{P} + \vec{P}_{ij}}_{\nabla\cdot\overleftrightarrow{P}\ \sim\ \text{tensor!}} \qquad ⑥$$

Second Order momentum: Energy equation.

~~See Chen's textbook's~~ (by taking moment $\frac{1}{2}m\vec{v}\vec{v}$).

$$\frac{\partial}{\partial t}\left(n_a\langle\tfrac{1}{2}mv^2\rangle\right) + \nabla\cdot\left(n_a\langle\tfrac{1}{2}m_a v_a^2\vec{v}_a\rangle\right)$$
$$= qn\vec{u}\cdot\vec{E} + R_a\cdot u_a + Q_a.$$

ABOVE are the single species equations. Add up all of the equations then we get the MHD equations:

$$\begin{cases} \frac{\partial\rho}{\partial t} + \vec{u}\cdot\nabla\rho = 0 \\ \rho_m\cdot\frac{D\vec{u}}{Dt} = \vec{J}\times\vec{B} - \nabla P_{tot} \end{cases} \qquad ⑦$$

We then talk about generalized ohm's law. (F.F. Chen, Chapter. 5.7)

Define:
$$\rho = n_i M + n_e m \approx n(M+m).$$
$$\vec{v} = \frac{1}{\rho}(n_i M\vec{v}_i + n_e m\vec{v}_e) \approx \frac{M\vec{v}_i + m\vec{v}_e}{M+m}.$$
$$\vec{j} = e(n_i\vec{v}_i - n_e\vec{v}_e) \approx ne(\vec{v}_i - \vec{v}_e) \qquad ⑧$$

for electrons, we can assume that we can ignore the electron inertia:

$$\Rightarrow\ m_e\cdot\frac{DU_e}{Dt} \simeq 0$$

collisions: $P_{en} + P_{ei} = [\nu_{en}\mu_{\nu}(\vec{u}_e - \vec{U}_n) + \vec{\nu}_{ei}\mu_{ei}(\vec{u}_e - \vec{U}_i)]n_e$ (mass)

Assume no viscousity, $\nabla\cdot\overleftrightarrow{P} = \nabla P_e$.

We get that:

$$0 = -en_e(\vec{E} + \vec{u_e} \times \vec{B}) - \nabla P_e + n_e m_e [\nu_{en}(\vec{u_n} - \vec{u_e})$$

$$+ \nu_{ei}(\vec{u_i} - \vec{u_e})]$$

(Under assumption, $\mu_{en} \simeq m_e$, $\mu_{ei} \simeq m_e$).

If $u_e \gg u_i$, $u_e \gg u_n$. (9)

Then $\vec{j_e} \simeq -e n_e \vec{u_e}$.

$$\Rightarrow \quad en_e \vec{E} + \nabla P_e = \vec{J_e} \times \vec{B} + \frac{m_e}{e}(\nu_{ei} + \nu_{en})\vec{j_e}$$

Divide it by $\frac{m_e}{e}(\nu_{ei} + \nu_{en})$

Define, $\sigma = \dfrac{e^2 n_e}{m_e(\nu_{en} + \nu_{ei})}$ (conducitivity) (10)

$\vec{\beta_e} = \dfrac{e\vec{B}}{m_e(\nu_{en} + \nu_{ei})}$ (Hall Parameter)

$\Rightarrow$ **Generalized Ohm's law:**

$$\sigma\left(\vec{E} + \frac{\nabla P_e}{en_e}\right) = \vec{J_e} + \vec{J_e} \times \vec{\beta_e}$$

At the Boundary we have:

$\vec{J_e} = 0$. $\quad \therefore \vec{E} = -\dfrac{\nabla P_e}{en_e}$ → the ambipolar E field

Consider that $\vec{E} = -\nabla\phi$. $\Rightarrow \nabla\phi = \dfrac{kT_e}{e} \cdot \dfrac{\nabla n_e}{n_e} \Rightarrow n_e = n_0 \exp\left(\dfrac{e\phi}{kT_e}\right)$

Boltzmann relation for electrons!

Let $\vec{E}^* = \vec{E} + \dfrac{\nabla P_e}{en_e}$,

$$\sigma\vec{E}^* = \vec{J_e} + \vec{J_e} \times \vec{\beta_e} \Rightarrow \sigma\vec{\beta_e} \times \vec{E}^* = \vec{\beta_e} \times \vec{J_e} + \vec{\beta_e} \times (\vec{J_e} \times \vec{\beta_e})$$

$$\vec{\beta_e} \times (\vec{J_e} \times \vec{\beta_e}) = \beta_e^2 \vec{J_e} - \vec{\beta_e}(\vec{\beta_e} \cdot \vec{J_e})$$

onsider $J_e \perp \vec{B}$. $\rightarrow \vec{B_e} \cdot \vec{J_e} = 0$.

$$\vec{J_e} \times \vec{\beta_e} = \beta_e^2 \vec{J_{e\perp}} - \sigma \vec{\beta_e} \times \vec{E}^*$$

$$\Rightarrow \begin{cases} \vec{J_{e\perp}} = \dfrac{\sigma}{1+\beta_e^2} \left(\vec{E_\perp}^* + \vec{\beta_e} \times \vec{E_\perp}^* \right) \\ \vec{J_{e\parallel}} = \sigma \vec{E_\parallel} \end{cases}$$

Application to ambipolar diffusion

Now we consider that the B equals to 0.

for ions we have:

$$m_i n \cdot \frac{D\vec{u_i}}{Dt} + \nabla P_i = e\vec{E} n + n \left[m_e \nu_{ie} (\vec{u_e} - \vec{u_i}) + \mu_{in} \nu_{in} (\vec{u_n} - \vec{u_i}) \right]$$

for electrons we have:

$$0 + \nabla P_e = -e\vec{E} n + m_e n_e \left[\nu_{ei} (\vec{u_i} - \vec{u_e}) + \nu_{en} (\vec{u_n} - \vec{u_e}) \right]$$

Add these two equations.

and note:

$n_e \nu_{ei} = n_i \nu_{ie}$, $n_e = n_i = n_0$ → drifting? fixed?

Assume $p = nKT$., if ignore ion inertia, $m_i \frac{Du_i}{Dt} \sim 0$.

$$\Rightarrow n(\vec{u_i} - \vec{u_e}) = -\frac{K(T_e + T_i)}{\mu_{in} \nu_{in}} \nabla n.$$

Diffusion & Mobility

⇓

Density gradient

Using the magnetic field to resist it.

First we consider the weakly ionized plasma without the B field. (fractional ionizations between 10^{-3} and 10^{-6})

Define :

collision frequency: $\nu = n_n \overline{\sigma v}$

Now go back to the fluid momentum equation.

$$mn \frac{D\vec{v}}{Dt} = \pm en\vec{E} - \nabla p - mn\nu\vec{v}$$

consider the problem as a steady state, that means

$$\frac{\partial \vec{v}}{\partial t} = 0$$

consider a quasi-linear condition, which means :

$\vec{v}$ is sufficiently small or ν is sufficiently large

then convection term will also vanish.

So we will get:

$$\vec{v} = \pm \frac{e}{m\nu}\vec{E} - \frac{KT}{m\nu} \cdot \frac{\nabla n}{n}$$

define $\mu \equiv \frac{|q|}{m\nu}$ ← mobility

define $D \equiv \frac{KT}{m\nu}$ (m^2/s) ↳ Diffusion.

Einstein relation:

$$\mu = \frac{|q| D}{KT}$$

$$\Gamma_j = n\vec{v_j} = \pm \mu_j n \vec{E} - D_j \nabla n.$$

Fick's law (special case)

$$\Gamma = -D\nabla n$$

[Not always valid in plasma, plasma has waves]

Ambipolar Diffusion. (a plasma created in a container. decays by diffusion to the walls)

$$\frac{\partial n}{\partial t} + \nabla \cdot \Gamma_j = 0,$$ in which the Γ_j is given by

he equation before.

We need to calculate the $\vec{E}$ field. Using the ndition that the $T_i = T_e = T$. we can get:

$$\Gamma = \mu_i n \vec{E} - D_i \nabla n = -\mu_e n \vec{E} - D_e \nabla n$$

$$\therefore \quad \vec{E} = \frac{D_i - D_e}{\mu_i + \mu_e} \cdot \frac{\nabla n}{n}.$$

In this way the

$$\Gamma = -\frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e} \nabla n$$

$$\therefore \quad D_a \equiv \frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e}$$ $\Rightarrow$ Ambipolar Diffusion Coefficient.

If we take $\mu_e \gg \mu_i$, then consider that $\vartheta \propto m^{-\frac{1}{2}}$,

We can get $\mu \propto m^{-\frac{1}{2}}$

$$\therefore \quad D_a \approx D_i + \frac{\mu_i}{\mu_e} D_e = D_i + \frac{T_e}{T_i} D_i$$

$$\therefore \quad D_a \approx 2 D_i$$

So the ambipolar electric field enhance the diffusion of ions by a factor of two.

For the problems we talked before, we assume there is no B field.
~~However~~ or in another word, parallel to the B field.

Now we talk about the diffusion across the B field.
(this can be explained by the collisions between particles)

In this way we can write a fluid motion equation for the perpendicular component

$$mn\cdot\frac{d\vec{v}_\perp}{dt} = \pm en(\vec{E} + \vec{v}_\perp \times \vec{B}) - KT\nabla n - mn\nu\vec{v} = 0$$

Again, we have the assumption that
the plasma is isothermal and $\frac{D\vec{v}_\perp}{Dt}$ can be neglect.

$$\therefore \quad mnv_x\nu = \pm enE_x - kT\cdot\frac{\partial n}{\partial x} \pm env_y B$$

$$mn\nu v_y = \pm enE_y - kT\cdot\frac{\partial n}{\partial x} \pm env_x B$$

$$\therefore \quad v_x = \pm\mu E_x - \frac{D}{n}\cdot\frac{\partial n}{\partial x} \pm \frac{\omega_c}{\nu}v_y$$

$$v_y = \pm\mu E_y - \frac{D}{n}\cdot\frac{\partial n}{\partial y} \mp \frac{\omega_c}{\nu}v_x$$

$\therefore$ We can solve the v_x and v_y from the equations above. ($\tau = \nu^{-1}$)

$$\therefore \quad v_y(1+\omega_c^2\tau^2) = \pm\mu E_y - \frac{D}{n}\cdot\frac{\partial n}{\partial y} - \omega_c^2\tau^2\cdot\frac{E_x}{B} \pm \omega_c^2\tau^2\cdot\frac{kT}{eB}\cdot\frac{1}{n}\cdot\frac{\partial}{\partial}$$

$$v_x(1+\omega_c^2\tau^2) = \pm\mu E_x - \frac{D}{n}\cdot\frac{\partial n}{\partial x} + \omega_c^2\tau^2\cdot\frac{E_y}{B} \mp \omega_c^2\tau^2\cdot\frac{kT}{eB}\cdot\frac{1}{n}\cdot\frac{\partial}{\partial}$$

Define some variables, which denote ~~the~~ $\vec{E} \times \vec{B}$ and ~~iagme~~ diamagnetic drifts.

$$v_{Ex} = \frac{E_y}{B}, \quad v_{Ey} = -\frac{E_x}{B}, \quad v_{Dx} = \mp \frac{kT}{eB} \cdot \frac{1}{n} \cdot \frac{\partial n}{\partial y}, \quad v_{Dy} = \pm \frac{kT}{eB} \cdot \frac{1}{n} \cdot \frac{\partial n}{\partial x}$$

Then we define:

$$\mu_\perp = \frac{\mu}{1 + \omega_c^2 \tau^2} \qquad D_\perp = \frac{D}{1 + \omega_c^2 \tau^2}$$

$$\therefore \quad \vec{v}_\perp = \underbrace{\pm \mu_\perp \vec{E} - D_\perp \frac{\nabla n}{n}}_{①} + \underbrace{\frac{\vec{v}_E + \vec{v}_D}{1 + (\nu^2 / \omega_c^2)}}_{②}$$

See, the first part means a parallel mobile motion like the occasion before. The second part means drift motion perpendicular to the gradients in potential and density., with modification from the drag factor: $1 + \frac{\nu^2}{\omega_c^2}$

Ambipolar diffusion across B.

$$\nabla \cdot \Gamma_i = \nabla_\perp \cdot (\mu_{i\perp} n \vec{E}_\perp - D_{i\perp} \nabla n) + \frac{\partial}{\partial z} \left(\mu_i n E_z - D_i \frac{\partial n}{\partial z} \right)$$

$$\nabla \cdot \Gamma_e = \nabla_\perp \cdot (-\mu_{e\perp} n \vec{E}_\perp - D_{e\perp} \nabla n) + \frac{\partial}{\partial z} \left(-\mu_e n E_z - D_e \cdot \frac{\partial n}{\partial z} \right)$$

Plasma. Resistivity:

ions: $M n \cdot \frac{d\vec{v}_i}{dt} = en(\vec{E} + \vec{v}_i \times \vec{B}) - \nabla P_i - \nabla \cdot \vec{\pi}_i + \vec{P}_{ie}$

electrons: $m n \cdot \frac{d\vec{v}_e}{dt} = -en(\vec{E} + \vec{v}_e \times \vec{B}) - \nabla P_e - \nabla \cdot \vec{\pi}_e + \vec{P}_{ei}$

$\vec{P}_{ie} = - \vec{P}_{ei}$ $\qquad \nabla \cdot \vec{\pi} \rightarrow$ Viscousity..

Writing the $\vec{P_{ei}}$ in terms of the collision frequency

$$\vec{P_{ei}} = mn\,(\vec{v_i} - \vec{v_e})\,\nu_{ei}$$

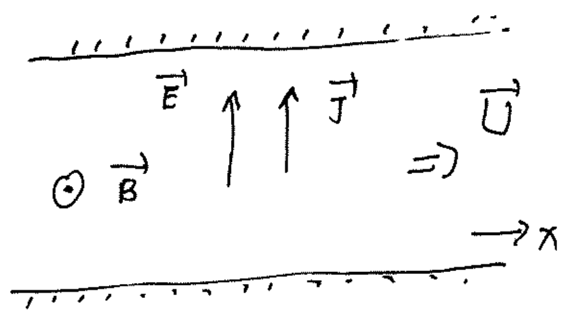
In another way,

$$\vec{P_{ei}} = \eta e^2 n^2\,(\vec{v_i} - \vec{v_e})$$

∴ specific resistivity

$$\nu_{ei} = \frac{ne^2}{m}\cdot \eta$$

* 1-D MHD channel Flow.



$$\frac{d}{dx}(\rho_m U) = 0 \qquad (1)$$

$$\rho_m U \cdot \frac{dU}{dx} = JB - \frac{dp}{dx} \qquad (2)$$

$$dq = de + p\cdot dv = C_v\, dT - \frac{p}{\rho_m^2}\cdot d\rho_m \qquad (3)$$

$$p = \rho_m R T.$$

(1) $\Rightarrow$ $\text{Flux} = \rho_m U = \text{const}, \quad \dfrac{d\rho_m}{dx} = -\dfrac{\rho_m}{U}\cdot\dfrac{du}{dx}$

(2) $\Rightarrow$ $\rho_m U^2 \cdot \dfrac{du}{dx} = -U\cdot\dfrac{dp}{dx} + UJB \qquad (4)$

(3) $\Rightarrow$ $\rho_m U\cdot\dfrac{dq}{dx} = \rho_m U\left(C_v\cdot\dfrac{dT}{dx} + \dfrac{p}{\rho_m^2}\cdot\dfrac{\rho_m}{U}\cdot\dfrac{du}{dx}\right) \qquad (5)$

(4) $\Rightarrow$ $\text{Flux}\cdot\dfrac{d}{dx}\left(\dfrac{U^2}{2}\right) = -U\cdot\dfrac{dp}{dx} + UJB \qquad (6)$

(5) $\Rightarrow$ $\text{Flux}\; C_v\cdot\dfrac{dT}{dx} = \text{Flux}\cdot\dfrac{dq}{dx} - \dfrac{p\,du}{dx} \qquad (7)$

⑥ + ⑦

$$Flux \frac{d}{dx}\left(\frac{U^2}{2} + C_v T\right) = Flux \cdot \frac{dq}{dx} + UJB - U \cdot \frac{dp}{dx} - p\frac{dU}{dx}$$

$$= Flux \cdot \frac{dq}{dx} + UJB - \frac{d}{dx}(pU)$$

If there is no other heat source,

$$Flux \cdot \frac{dq}{dx} = \frac{J^2}{\sigma}$$

Additionally,

$$\frac{d}{dx}(pU) = Flux \cdot \frac{d}{dx}(RT)$$

Hence.

$$⑥ + ⑦ \Rightarrow \quad Flux \cdot \frac{d}{dx}\left[\frac{U^2}{2} + \underbrace{C_v T + RT}_{C_p T}\right] = \frac{J^2}{\sigma} + UJB$$

$$W = \vec{E} \cdot \vec{J}$$

$$= \vec{E'} \cdot \vec{J} + (\vec{J} \times \vec{B}) \cdot \vec{u}$$

$$\vec{E'} \cdot \vec{J} = \frac{1}{\sigma}(\vec{J} + \vec{J} \times \vec{\beta}) \cdot \vec{J} = \frac{J^2}{\sigma}$$

$$\therefore W = \frac{J^2}{\sigma} + (\vec{J} \times \vec{B}) \cdot \vec{u}$$

$$\therefore \vec{E} \cdot \vec{J} = Flux \cdot \frac{d}{dx}\left[\frac{u^2}{2} + C_p T\right]$$

Ionization Gas

Saha Equation

$$\frac{n_i}{n_n} \simeq 2.4 \times 10^{21} \cdot \frac{T^{\frac{3}{2}}}{n_i} \cdot e^{-\frac{U_i}{kT}}$$ ionization energy

Plasma Wave theory.

- Linearize Theory

A general ~~wave~~ form for wave.

$$n = \bar{n} \exp[i(\vec{k} \cdot \vec{r} - \omega t)]$$

where in Cartesian Coordinates,

$$\vec{k} \cdot \vec{r} = k_x x + k_y y + k_z z$$

$\vec{k}$ is called the propagation constant.

In this part we only need to talk about the 1-D. case. Assume the wave propagates along x direction.

$$n = \bar{n} \exp[i(kx - \omega t)]$$

We define

$$\frac{dx}{dt} = \frac{\omega}{k} \equiv v_\varphi$$, phase velocity. (Ok to be faster th

$$v_g = \frac{d\omega}{dk}.$$ Local Oscillation. propagation.

We call $\omega(k)$ dispersion relations.

We use $\vec{g}_1$ to denote the $g_1 \exp[i(\vec{k} \cdot \vec{r} - \omega t)]$

Fourier Transform

$$-i\omega \rightleftharpoons \frac{\partial}{\partial t}$$

$$i\vec{k} \rightleftharpoons \nabla$$

$$\vec{k} \times \vec{t} \rightleftharpoons \nabla \times t$$

$$i\vec{k} \cdot \vec{f} \rightleftharpoons \nabla \cdot f$$

Plasma Oscillation

① conditions description:

electrons displaced from a uniform background of ions

characteristic frequency, " Plasma frequency "

②. pre-conditions:

1. No magnetic field. $(B=0)$
2. No thermal motions $(KT=0)$
3. ions are fixed in space in a uniform distribution.
4. plasma is infinite in extent
5. electron motions occur only in x directions

③ Equations we need to use:

$$m n_e \left[\frac{\partial \vec{v_e}}{\partial t} + (\vec{v_e} \cdot \nabla) \vec{v_e} \right] = -e n_e \vec{E}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{v_e}) = 0$$

When the electron inertia is important, the deviation from neutrality is the main effect in this particular case.

then

$$\varepsilon_0 \nabla \cdot \vec{E} = e(n_i - n_e)$$

$n_e = n_0 + n_1$, $\vec{v_e} = \vec{v_0} + \vec{v_1}$ $\vec{E} = \vec{E_0} + \vec{E_1}$

Consider the ~~pre~~ assumptions,

$$\nabla n_0 = \vec{v_0} = \vec{E_0} = 0$$

$$\frac{\partial n_0}{\partial t} = \frac{\partial \vec{v_0}}{\partial t} = \frac{\partial E_0}{\partial t} = 0$$

$$m \cdot \left[\frac{\partial \vec{v_1}}{\partial t} + (\vec{v_1} \cdot \nabla)\vec{v_1} \right] = -e\vec{E_1}$$

$$\frac{\partial n_1}{\partial t} + \nabla \cdot (n_0 \vec{v_1} + n_1 \vec{v_1}) = 0$$

Note $(\vec{v_1} \cdot \nabla)\vec{v_1}$ is a second order small perturbation.

So we ignore it. and $\vec{v_1} \cdot \nabla n_0 = 0$ (assumptions)

Poisson Equation: $\varepsilon_0 \nabla \cdot \vec{E_1} = -en_1$

$$\therefore \omega^2 = n_0 e^2 / m\varepsilon_0$$

$$\therefore \omega_p = \left(\frac{n_0 e^2}{\varepsilon_0 m} \right)^{\frac{1}{2}} \text{ rad/sec}$$

$$\therefore \omega_p / 2\pi \approx 9\sqrt{n} \quad (n \text{ in } m^{-3})$$

Note that as the Group Velocity $= 0$, there will be no information propagating.

Electron Plasma Waves, (Langmuir Wave)

If we want to make the plasma oscilliation propagate,

Then we need thermal motion.

To describe the thermal motion under fluid description, we need to add a term $-\nabla P_e$

In our one dimensional case, we thus have:

$$\nabla P_e = 3KT_e \nabla n_e = 3KT_e \nabla (n_0 + n_1) = 3KT_e \cdot \frac{\partial n_1}{\partial x} \hat{x}$$

$\therefore$ the linearized equation of motion is

$$m n_0 \cdot \frac{\partial v_1}{\partial t} = -e n_0 E_1 - 3KT_e \cdot \frac{\partial n_1}{\partial x}$$

thus we have that

$$\omega^2 = \omega_p^2 + \frac{3}{2} k^2 v_{th}^2, \quad \text{where} \quad v_{th}^2 \equiv \frac{2KT_e}{m}$$

$$v_{group} = \frac{3}{2} \cdot \frac{v_{th}^2}{v_\phi}$$

Ion acoustic waves

In absence of collisions, ordinary acoustic waves would not occur. Ions can still transmit vibration through the effect of $\vec{E}$ field.

Because of the mass of ions is so big so we consider is as a low frequency effection.

We therefore assume $n_i = n_e = n$

Thus.

$$Mn \left[\frac{\partial \vec{v}_i}{\partial t} + (\vec{v}_i \cdot \nabla) \vec{v}_i \right] = en\vec{E} - \nabla p = -en\nabla\phi - \gamma_i K T_i \nabla n.$$

for electrons, we consider that

$$n_e = n = n_0 \exp\left(\frac{e\phi_1}{kT_e}\right) = \dots \text{ (Taylor Expansion)}.$$

so the perturbation of density of electrons,

$$n_1 = n_0 \cdot \frac{e\phi_1}{kT_e}$$

Assume $\phi_0 = 0$. as $\vec{E_0} = 0$

$$\therefore \omega^2 = k^2 \left(\frac{KT_e}{M} + \frac{\gamma_i KT_i}{M}\right) \qquad \therefore \frac{\omega}{K} = \left(\frac{KT_e + \gamma_i KT_i}{M}\right)^{\frac{1}{2}} \equiv v_s$$

ion waves are basically constant velocity waves and exist only when there are thermal motions.

When $T_i \ll T_e$,

$$v_s = \sqrt{\frac{KT_e}{M}}.$$

Now let us validate the assumption that $n_i = n_e$.. We use the linearize Poisson equation..

$$\varepsilon_0 \nabla \cdot \vec{E_1} = \varepsilon_0 k^2 \phi_1 = e(n_{i1} - n_{e1})$$

in which, $n_{e1} = \frac{e\phi_1}{kT_e} \cdot n_0$, $n_{i1} = \frac{k}{\omega} n_0 v_{i1}$

So in this way we get that,

$$\frac{w}{k} = \left(\frac{KT_e}{M} \frac{1}{1+k^2\lambda_D^2} + \frac{\gamma_i KT_i}{M}\right)^{\frac{1}{2}}$$

$k^2\lambda_D^2 = \left(\frac{2\pi\lambda_D}{\lambda}\right)^2$. In this way we can figure out that except sheath, the "$n_i = n_e$" assumption is valid.

Electrostatic Electron Oscillation Perpendicular to B

This time let us consider the plasma waves under finite $\vec{B}$ ield.

Assume :

1. ions are fixed and uniform.
2. No thermal motions, $KT_e = 0$.
3. n_0 = constant, $\vec{B_0}$ = constant.
   $\vec{E_0} = 0$, $\vec{v_0} = 0$

linearized Equations:

$$m \cdot \frac{\partial \vec{v_{e1}}}{\partial t} = -e\left(\vec{E_1} + \vec{v_{e1}} \times \vec{B_0}\right).$$

$$\frac{\partial n_{e1}}{\partial t} + n_0 \nabla \cdot \vec{v_{e1}} = 0$$

$$\varepsilon_0 \nabla \cdot \vec{E_1} = -e n_{e1}$$

consider longitudinal waves with $\vec{k} \parallel \vec{E}$

Set that $\vec{k} = k\hat{x}$, $\vec{E} = E\hat{x}$

$$\therefore \ \omega^2 = \omega_p^2 + \omega_c^2$$

ω_c: clcytron frequency

ω_p: plasma frequency (usually, electron ~~fre~~ oscillation frequency).

electrostatic ion cyclotron

ion acoustic wave when $\vec{k}$ is perpendicular to $\vec{B}_0$

We shall let $\vec{k}$ be almost perpendicular to $\vec{B}_0$

Assume :

1. $\vec{n}_0$, $\vec{B}_0$ constant and uniform
2. $\vec{v}_0 = \vec{E}_0 = 0$, $T_i = 0$
3. $\vec{k} \times \vec{E} = 0$, $\vec{E} = -\nabla\phi$

Equations :

$$M \cdot \frac{\partial \vec{v}_{i1}}{\partial t} = -e\nabla\phi_1 + e\,\vec{v}_{i1} \times \vec{B}_0$$

$$\therefore -i\omega M v_{ix} = -eik\phi_1 + e v_{iy} B_0$$

$$-i\omega M v_{iy} = -e v_{ix} B_0$$

$$\therefore v_{ix} = \frac{ek}{M\omega}\phi_1 \left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1}$$

in which $\Omega_c = \frac{eB_0}{M}$

Using plasma approximation,

$$n_i = n_e$$

$$\therefore \omega^2 = \Omega_c^2 + k^2 v_s^2 \quad (kT_i = 0)$$

Low Hybrid Frequency

Now consider what happens when θ is exactly $\frac{\pi}{2}$

In this way the electrons are not allowed to preserve change neutrality by flowing along the lines of force.

In this way we ~~need to use Poisson Equation~~

are not allowed to obey Boltzmann Equation.

So $$v_{ix} = \frac{ek}{M\omega}\phi_1 \left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1}$$

$$v_{ex} = -\frac{ek}{m\omega}\phi_1 \left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$$

if ($n_i = n_e$) directly used, then

$$\omega = \left(\Omega_c \omega_c\right)^{\frac{1}{2}} \equiv \omega_{lh}$$

In very low frequency plasmas,

$$\frac{1}{\omega_l^2} = \frac{1}{\omega_c \Omega_c} + \frac{1}{\Omega_p^2}$$

Light Waves propagated in plasmas.

ht

ave

$$\nabla \times \vec{E}_1 = -\dot{B}_1$$

$$C^2 \nabla \times \vec{B}_1 = \dot{E}_1$$

In Vacuum. $\vec{j} = 0$

$\varepsilon_0 \mu_0 = \frac{1}{c^2}$

$\therefore$ ① $\nabla \times \dot{\vec{E}}_1 = -\ddot{\vec{B}}_1$, using Fourier Transform.

$$\therefore \omega^2 \vec{B}_1 = -c^2 [\vec{k}(\vec{k}\cdot\vec{B}_1) - k^2 \vec{B}_1] ,\quad \vec{k}\cdot\vec{B}_1 = 0$$

$$\therefore \omega^2 = k^2 c^2$$

Now we come into plasma.

Because of the first-order changed particle motions.

we have:

$$C^2 \nabla \times \vec{B}_1 = \frac{\vec{J}_1}{\varepsilon_0} + \dot{\vec{E}}_1$$

By transverse waves, $\vec{k} \cdot \vec{E_1} = 0$

$$\therefore \quad (\omega^2 - c^2 k^2) \vec{E_1} = -i\omega \vec{j_1} / \varepsilon_0$$

Compared to the last section, we got an extra term "$-i\omega \vec{j_1}/\varepsilon_0$" because of the propagation of the waves in plasma.

$\vec{j_1}$ comes from electron motion:

$$\vec{j_1} = -n_0 e \vec{v_{e1}}$$

From lineralized electron equation ~~for~~ of motion,

$$\vec{v_{e1}} = \frac{e\vec{E_1}}{im\omega}$$

$$\therefore \quad \underline{\omega^2 = \omega_p^2 + c^2 k^2}, \qquad k = \frac{1}{c}\sqrt{\omega^2 - \omega_{pe}^2}$$

$n \uparrow \rightarrow \omega_{pe} \uparrow \rightarrow k \downarrow \rightarrow \lambda \uparrow$ when

$W = W_{pe}$, n_c can be defined.

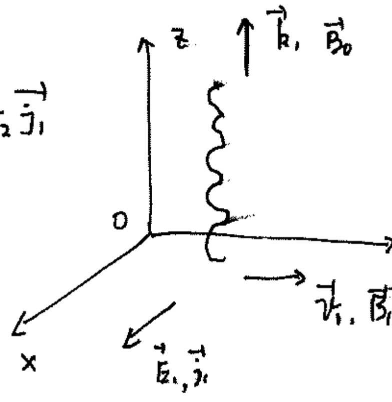
if $n > n_c$, then Blackout!

Alfvén Wave.

low frequency ion oscilliation with propagation along $\vec{B_0}$

From Maxwell's equation we have:

$$\nabla \times \nabla \times \vec{E_1} = -\vec{k}(\vec{k} \cdot \vec{E_1}) + k^2 \vec{E_1} = \frac{\omega^2}{c^2}\vec{E_1} + \frac{i\omega}{\varepsilon_0 c^2}\vec{j_1}$$



As $\vec{k} = k\hat{z}$, $\vec{E}_1 = E_1\hat{x}$. only x component is nontrivial.

$$\therefore \quad \varepsilon_0 (\omega^2 - c^2k^2) \vec{E}_1 = -i\omega n_0 e (v_{ix} - v_{ex})$$

As, $v_{ix} = \frac{ie}{M\omega} \left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1} E_1$

$$v_{iy} = \frac{e}{M\omega} \cdot \frac{\Omega_c}{\omega} \left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1} E_1$$

Since. $\omega_c^2 \gg \omega^2$.

$$v_{ex} = \frac{ie\,\omega^2}{m\omega \cdot \omega_c^2} E_1 \to 0$$

$$v_{ey} = -\frac{e}{m} \cdot \frac{\omega_c}{\omega^2} \cdot \frac{\omega^2}{\omega_c^2} E_1 = -\frac{E_1}{B_0}$$

$$\therefore \quad \omega^2 - c^2k^2 = \Omega_p^2 \left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1}$$

if $\omega^2 \ll \Omega_c^2$

$$\frac{\omega^2}{k^2} = \frac{c^2}{1 + (\rho\mu_0/B_0^2)c^2}$$

$$\therefore \quad \frac{\omega}{k} = v_\phi = \frac{B_0}{(\mu_0\rho)^{\frac{1}{2}}} = v_A \quad \text{Alfvén velocity}$$

$$= \frac{c}{\sqrt{1 + \frac{4\pi\rho mc^2}{B_0^2}}}$$

Two Stream Instability

Conditions: uniform plasma in which the ions are stationary and the electrons have a velocity $\vec{V_0}$

Plasma : cold ($KT_e = KT_i = 0$). $B_0 = 0$.

So the linearized equations are:

$$M n_0 \cdot \frac{\partial \vec{v_{i1}}}{\partial t} = e n_0 \vec{E_1}$$

$$m n_0 \left[\frac{\partial \vec{v_{e1}}}{\partial t} + (\vec{v_0} \cdot \nabla) \vec{v_{e1}} \right] = - e n_0 \vec{E_1}.$$

Assume. $\vec{E_1} = E \cdot e^{i(kx - wt)} \hat{x}$

$\therefore -iwMn_0 \vec{v_{i1}} = e n_0 \vec{E_1}$. $\vec{v_{i1}} = \frac{ie}{Mw} E \hat{x}$

$\therefore mn_0(-iw + ikv_0)\vec{v_{e1}} = -en_0\vec{E_1}$, $\vec{v_{e1}} = -\frac{ie}{m} \cdot \frac{E\hat{x}}{w - kv_0}$

The ion equation of continiuity yields:

$$\frac{\partial n_{i1}}{\partial t} + n_0 \nabla \cdot \vec{v_{i1}} = 0, \quad n_{i1} = \frac{ien_0 k}{Mw^2} E$$

The electron equation of continuity is:

$$\frac{\partial n_{e1}}{\partial t} + n_0 \nabla \cdot \vec{v_{e1}} + (\vec{v_0} \cdot \nabla) n_{e1} = 0$$

$$\therefore n_{e1} = \frac{kn_0}{w - kv_0} v_{e1} = -\frac{iekn_0}{m(w - kv_0)^2} E.$$

Since the instabilities are high-frequency plasma oscillations, we may not use the plasma approximation, but use

$$\varepsilon_0 \cdot \nabla \cdot \vec{E_1} = e(n_{i1} - n_{e1}).$$

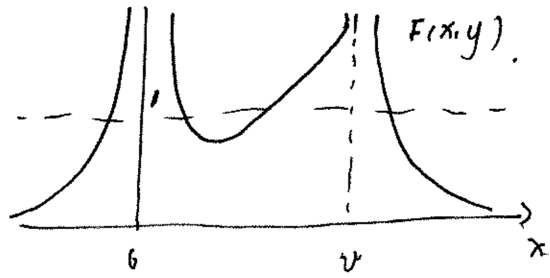
$\therefore$ the dispersion relationship:

$$1 = w_p^2 \left[\frac{m/M}{w^2} + \frac{1}{(w - kv_0)^2} \right]$$

The dispersion relationship should have four roots. (Fourth order equations)

define $x = \frac{\omega}{\omega_p}$, $y = \frac{k v_0}{\omega_p}$

$$\therefore \quad 1 = \frac{m/M}{x^2} + \frac{1}{(x-y)^2} \equiv F(x,y)$$



For a given k, v_0 has to be sufficiently small.

(no physics sense, as the v is the source of energy, when v is small, not enough energy for instabilities). Why?

If $\omega_{root} = \alpha_j + i\gamma_j$

$$\therefore \vec{E}_1 = \underbrace{E e^{i(kx - \alpha_j t)}}_{\text{expotentially Decay...}} \underbrace{e^{\gamma_j t}}_{\text{expotentially growth.}} \hat{x}$$

We use isothermal assumption which is not true.

See. F.F. chen's P201. for more information.